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Vector Image Method for the Derivation of Elastostatic Solutions for Point Sources in a Plane Layered Medium

Part I: Derivation and Simple Examples

By

Nabil Fares
Graduate student

Victor C. Li
Associate Professor of Civil Engineering
Member of ASME

Department of Civil Engineering
Massachusetts Institute of Technology
Cambridge Ma. 02139

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Abstract

This paper presents an image method algorithm for the derivation of elastostatic solutions for point sources in bonded halfspaces assuming the infinite space point source is known. Specific cases have been worked out and shown to coincide with well known solutions in the literature.

Introduction

Point sources (Green's functions are point sources) for some given (linear) governing differential equations and boundary conditions are important because of two main reasons. First, any localized process when viewed from a sufficient distance can be modelled as some suitably chosen point sources. Second, Green's functions can be used to reframe the governing differential equations and boundary conditions in an integral equation form; the integral equation form can, for example, be used as the basis for numerically analyzing a large class of problems using the boundary element method.

This paper presents a new algorithm for the derivation of point sources of elastostatics in bonded halfspaces assuming the point source in infinite space is known. The method is similar to the image method that is familiar when deriving Green's functions in plane layered media where there is only one unknown scalar field in the governing equations such as in heat conduction, potential flow and electrostatics problems. In a sequel paper, the algorithm is then used to formally derive new Green's functions for any point source in a region consisting of an elastic layer perfectly bonded to two elastic halfspaces. Numerical solutions for the displacement fields of nucleii of strain in an elastic plate are also presented in the sequel paper.

Background

There are many known Green's functions for halfspace problems in elastostatics. Most of the known Green's functions are specialized for a single halfspace having a stress free surface (a special case of bonded elastic halfspaces when one of the regions has zero rigidity). Some of these known solutions are briefly surveyed with occasional comments on the method of derivation.

The most used point source solutions are the point force, the dislocation and the nuclei of strain (or double couple) solutions. The point force solution for 2-D plane problems in a halfspace with a free surface (Mellan 1932), 3-D problem in a halfspace with a free surface (Mindlin 1936) and 3-D problem in bonded elastic halfspaces (Rongved 1955) are known. Rongved obtained the Green's function through the use of the Papkovich-Neuber potentials and arguments from harmonic analysis; the resulting solution is in the form of the sum of a point force solution in infinite space and some point sources at the image point with respect to the interface plane.

The elastic fields of screw and edge dislocations in bonded halfspaces were first given by Head (1953 a,b). The screw dislocation problem is obtained by the method of images (since there is only one field variable).

There are six nucleii of strain sources. The solution to the first (double couple in a plane parallel to the free surface) was given by Steketee (1958), the remaining five sources were given by Maruyama (1964). Maruyama used image nucleii of strain sources to cancel the tangential component of the surface traction on the free surface. He then used the Boussinesq solution (in Galerkin vector representation) and the remaining normal tractions on the free surface in a Hankel/Fourier transformed space to obtain the rest of the fields afterwhich he transformed the solution back to real space. This procedure is highly specific to half space problems with a free surface and cannot be generalized to multiple layered systems.

Finally, we note the existence of an image method for perfectly bonded elastic halfspaces in terms of the Papkovich-Neuber potentials (Aderogba 1977). Aderogba presented the algorithm for obtaining the four image potentials which involves multiple integrations with respect to the coordinate perpendicular to the interface plane and differentiation with respect to all three coordinates. The algorithm based on the Hansen potentials presented in this paper involves 3 potentials only, and only differentiation of the (infinite space) potentials with respect to the coordinate perpendicular to the interface plane (as well as multiplication by scalars) is required to obtain the image potentials. This distinction is especially important when the image algorithm is repeatedly applied to obtain the fields due to point sources in regions consisting of an elastic layer perfectly bonded to two elastic halfspaces.

Preliminary Considerations

The image method presented in this paper is dependent upon expressing the displacements in terms of potentials. The specific potentials employed are the analogue to Hansen's potentials for elastostatics and dynamics. Unlike Ben-Menahem and Singh (1968) the potentials are not expanded in terms of eigenfunctions; instead the algorithm operates directly on the potentials. Note however, that the eigenfunction expansion technique was used in the derivation of the algorithm (see Appendix 2).

Specifically, we express the displacement field is expressed in terms of the Hansen potentials φ_1, φ_2 and φ_3 in the following manner:

$$\underline{u}(h,\delta,\underline{\varphi}_R,\underline{\varphi}_L) = \underline{N}(h,\varphi_1) + \underline{F}(\delta,h,\varphi_2) + \underline{M}(h,\varphi_3)$$

$$\underline{\varphi}_R = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \qquad \underline{\varphi}_L = \begin{bmatrix} \varphi_3 \end{bmatrix}$$

$$\underline{N}(h,\varphi_1) = \nabla\varphi_1(x,y,z-h)$$

$$\underline{F}(\delta,h,\varphi_2) = + 2\cdot\hat{e}_z\cdot\frac{\partial}{\partial z}\varphi_2(x,y,z-h) \tag{1}$$

$$- \nabla\varphi_2(x,y,z-h)$$

$$- 2\cdot\delta\cdot(z-h)\ \nabla\frac{\partial}{\partial z}\varphi_2(x,y,z-h)$$

$$\underline{M}(h,\varphi_3) = \nabla \times [\hat{e}_z\cdot\varphi_3(x,y,z-h)]$$

where: ∇ is the gradient operator

$\nabla \times$ is the curl operator

$\nabla^2\varphi_1 = \nabla^2\varphi_2 = \nabla^2\varphi_3 = 0$

$\delta = \dfrac{\lambda + \mu}{\lambda + 3\mu}$

λ is the Lame constant

μ is the shear modulus

h is a scalar for shifting the z-coordinate

Note that the potentials φ_1, φ_2 and φ_3 have to be harmonic in order for $\underline{N}$, $\underline{F}$ and $\underline{M}$ to satisfy equilibrium. The Cartesian components for the displacements, strains and stresses are given in appendix 1 of Fares and Li (1986). It is shown in appendix 3 that $\underline{\varphi}_L$ is associated with the antiplane mode of deformation.

In order for these potentials to be useful, a method to obtain these potentials given an elastic field satisfying equilibrium is described below. Note that:

$$\nabla\cdot\underline{N} = \nabla\cdot\underline{M} = 0 \qquad\qquad \nabla\cdot\underline{F} = 2\cdot(1-\delta)\cdot\frac{\partial^2\varphi_2}{\partial z^2}$$

$$\nabla \times \underline{N} = 0 \tag{2}$$

$$\nabla \times \underline{F} = 2\cdot(1+\delta)\cdot\frac{\partial^2\varphi_2}{\partial y\partial z}\cdot\hat{e}_x - 2\cdot(1+\delta)\cdot\frac{\partial^2\varphi_2}{\partial x\partial z}\cdot\hat{e}_y$$

$$\nabla \times \underline{M} = \frac{\partial^2\varphi_3}{\partial x\partial z}\cdot\hat{e}_x + \frac{\partial^2\varphi_3}{\partial y\partial z}\cdot\hat{e}_y + \frac{\partial^2\varphi_3}{\partial z^2}\cdot\hat{e}_z$$

Hence for a given displacement field $\underline{u}$, the following may be calculated:

$$\nabla\cdot\underline{u} = 2\cdot(1-\delta)\cdot\frac{\partial^2\varphi_2}{\partial z^2} \tag{3}$$

and

$$(\nabla \times \underline{u})\cdot\hat{e}_z = \frac{\partial^2\varphi_3}{\partial z^2} \tag{4}$$

The potentials φ_2 and φ_3 may be obtained by integrating (3) and (4):

$$\varphi_2 = \int dz \int dz \frac{\nabla\cdot\underline{u}}{2\cdot(1-\delta)} + z\cdot F_2(x,y) + G_2(x,y)$$

$$\varphi_3 = \int dz \int dz [(\nabla \times \underline{u})\cdot\hat{e}_z + z\cdot F_3(x,y) + G_3(x,y) \tag{5}$$

The integration constants F_i's and G_i's are chosen such that

φ_2 and φ_3 are harmonic in the required region. In addition, all singularities of the potentials must be in the region where the source occurs. This is made clearer in appendix 3 when we consider examples of the use of the algorithm. Finally, once φ_2 and φ_3 are determined, whatever remains in the displacement field (see equation 1) is ascribed to φ_1. If the given displacement field does satisfy equilibrium, the field should be expressible in terms of these three potentials (see Ben-Menahem and Singh 1968, and Morse and Feshbach 1953).

The Hansen potentials for a point force, and a line force perpendicular to the z-direction are given in Appendix 1.

<u>The Algorithm</u>

The algorithm and the notation associated with it can now be described. Consider two elastic halfspaces perfectly bonded along an interface plane at z=0 (see figure 1). The material properties of region 1 are described by μ_1 and δ_1, and of region 2 by μ_2 and δ_2. Next we define the following:

$$\bar{\varphi}(x,y,z) \equiv \varphi(x,y,-z)$$

$$\gamma \equiv \mu_2/\mu_1 \tag{6}$$

$$a \equiv \frac{(\delta_1+1)}{(\delta_1+\gamma)} \qquad\qquad b \equiv \frac{(\delta_1+1)}{(\gamma\cdot\delta_2+1)}$$

Note that if φ is harmonic then $\bar{\varphi}$ is also harmonic and hence can be used as a Hansen potential for $\underline{N}$, $\underline{F}$ and $\underline{M}$.

The algorithm states that if we have the representation for a point source in infinite space of elastic constants similar to those of region 1 at the location $x=y=0$ and $z=h$ described by the displacement field:

$$\underline{u}^0 \equiv \underline{u}^0(h,\delta_1,\underline{\varphi}_R^0,\underline{\varphi}_L^0) \tag{7}$$

then the displacement fields in regions 1 and 2 for a similar point source in region 1 at $x=y=0$ and $z=h$ are given by:

$$\begin{aligned} \underline{u}^1 &= \underline{u}^0 + \underline{u}(-h,\delta_1,\bar{\underline{\varphi}}_R^1,\bar{\underline{\varphi}}_L^1) \\ \underline{u}^2 &= \underline{u}(h,\delta_2,\underline{\varphi}_R^2,\underline{\varphi}_L^2) \end{aligned} \tag{8}$$

where the image potentials may be obtained from the source potentials with the following operations:

$$\underline{\varphi}_R^1 = \underline{\underline{R}}_R(-h,a,b,\delta_1)\cdot\underline{\varphi}_R^0$$

$$\underline{\varphi}_L^1 = \underline{\underline{R}}_L(\gamma)\cdot\underline{\varphi}_L^0$$

$$\underline{\varphi}_R^2 = \underline{\underline{T}}_R(h,a,b,\delta_2,\delta_1)\cdot\underline{\varphi}_R^0$$

$$\underline{\varphi}_L^2 = \underline{\underline{T}}_L(\gamma)\cdot\underline{\varphi}_L^0$$

$$\underline{\underline{R}}_R(-h,a,b,\delta_1) \equiv \left[\begin{array}{c|c} -2\delta_1(1-a)h\cdot\frac{\partial}{\partial z} & +(1-b) - 4\delta_1^2(1-a)h^2\cdot\frac{\partial^2}{\partial z^2} \\ \hline +(1-a) & +2\delta_1(1-a)h\cdot\frac{\partial}{\partial z} \end{array}\right]$$

$$\underline{\underline{T}}_R(h,a,b,\delta_2,\delta_1) \equiv \left[\begin{array}{c|c} +a & -2(\delta_2 b-\delta_1 a)h\cdot\frac{\partial}{\partial z} \\ \hline 0 & +b \end{array}\right]$$

$$\underline{\underline{R}}_L(\gamma) \equiv \left[\frac{1-\gamma}{1+\gamma} \right] \qquad \underline{\underline{T}}_L(\gamma) \equiv \left[\frac{+2}{1+\gamma} \right] \tag{9}$$

The above operators are denoted by 'R' and 'T' and stand for 'Reflection' and 'Transmission' operators respectively, in analogy with wave reflection and transmission operators for plane waves in elastodynamic problems.

We note that:

$$\bar{\underline{P}}_R^1 = \overline{\underline{\underline{R}}_R(-h,a,b,\delta_1)\cdot\underline{P}_R^0} = \bar{\underline{\underline{R}}}_R(-h,a,b,\delta_1)\cdot\bar{\underline{P}}_R^0$$

and:

$$\bar{\underline{\underline{R}}}_R(-h,a,b,\delta_1) = \underline{\underline{R}}_R(+h,a,b,\delta_1) \tag{10}$$

Note that the $\underline{P}_L^{1,2}$ are simple multiplicatives of $\underline{P}_L^0$. The case when $\underline{P}_R^0 = 0$ corresponds to the purely anti-plane problem, and thus, the algorithm reduces to the scalar image method for that case.

The derivation of the above algorithm is given in Appendix 2, and some sample known solutions are rederived in appendix 3; namely the screw dislocation in a half space with a free surface and Mindlin's solution of a point force interior to a halfspace.

Conclusions and further recommendations

A vector image method has been presented, for elastic problems with planar interfaces. An algorithm has been presented on how to derive point source solutions for two bonded elastic halfspaces. Specific cases have been worked out (in appendix 3) and shown to coincide with well known solutions in the literature. Further details and sample cases could be found in Fares and Li (1986).

The method of deriving the algorithm suggests that an analogous algorithm can be obtained for spherical interface problems, and 2-D (but not 3-D) cylindrical interface problems in elastostatics. This suggestion is supported by the existence of a scalar image method and Hansen potential representations for both these geometries.

Finally, it would also be of interest to investigate equivalent algorithms for other governing equations. For example, elastodynamics and poroelasticity could be potential candidates for such an investigation. Elastodynamic problems, in particular, do have Hansen potential representations that have been well established and used and could be investigated first without the considerable preliminary formulations that are needed for poroelastic problems.

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Appendix 1: Sample potentials for some point sources

Point Force:

The displacement field due to a point force at the origin can be written as:

$$u_i = \frac{1}{4\pi\mu\cdot(1+\delta)}\cdot\left[p_i\frac{1}{r} + \delta\cdot p_k\cdot x_k\cdot x_i\cdot\frac{1}{r^3} \right] \tag{1.1}$$

where: $\delta \equiv \frac{\lambda+\mu}{\lambda+3\mu} \equiv \frac{1}{\kappa} \equiv \frac{1}{3-4\nu}$

$r^2 = x^2 + y^2 + z^2$

p_i are the magnitudes of the point forces in the i^{th}-direction

The Hansen potentials for the point force can be obtained by using (1) and (5):

$$\varphi_1 = \frac{\beta}{2}\cdot\left[p_1\cdot\frac{x}{r\pm z} + p_2\cdot\frac{y}{r\pm z} \pm p_3\cdot\ln(r\pm z) \right]$$

$$\varphi_2 = \frac{\beta}{2}\cdot\left[-p_1\cdot\frac{x}{r\pm z} - p_2\cdot\frac{y}{r\pm z} \pm p_3\cdot\ln(r\pm z) \right] \tag{1.2}$$

$$\varphi_3 = \beta\cdot\left[p_1\cdot(1+\delta)\cdot\frac{y}{r\pm z} - p_2\cdot(1+\delta)\cdot\frac{x}{r\pm z} \right]$$

where: $\beta \equiv 1/[4\pi\mu\cdot(1+\delta)]$.

Note that if the upper (lower) "sign" of ± is chosen in one expression, the upper (lower) "signs" must be chosen throughout for all the potentials. Also note that taking r+z (r-z) in the expressions makes the potentials (but not necessarily the displacements) singular when x=y=0 and z<0 (z>0).

Line forces at x = 0 acting on the z = 0 plane

The displacement field due to a line force can be written (for plain strain) as:

$$u_i = \frac{\alpha}{4\pi\mu\delta}\cdot\left[-p_i\cdot\ln\xi + \delta\cdot p_k\cdot x_k\cdot x_i\cdot\frac{1}{\xi^2}\right] \qquad \text{for } i,k = 1,3$$

and

$$u_2 = 0 \tag{1.3}$$

where: $\alpha \equiv \frac{\lambda+\mu}{\lambda+2\mu}$ $\qquad \delta \equiv \frac{\lambda+\mu}{\lambda+3\mu}$

$\xi^2 = x^2 + z^2$

p_1 and p_3 are the magnitude of the line forces

The Hansen potentials for the line force can be obtained by using (1) and (5):

$$\varphi_1 = \frac{\alpha}{8\pi\mu\delta}\cdot\Big[\; p_1\cdot\left[z\cdot\arctan(\tfrac{z}{x}) - x\cdot\ln\xi + (1+\delta)\cdot x\right] - p_3\cdot\left[z\cdot\ln\xi - z + x\cdot\arctan(\tfrac{z}{x})\right]\Big]$$

$$\varphi_2 = \frac{\alpha}{8\pi\mu\delta}\cdot\Big[-p_1\cdot\left[z\cdot\arctan(\tfrac{z}{x}) - x\cdot\ln\xi + (1+\delta)\cdot x\right] - p_3\cdot\left[z\cdot\ln\xi - z + x\cdot\arctan(\tfrac{z}{x})\right]\Big]$$

$$\varphi_3 = 0 \tag{1.4}$$

Dislocations parallel to the z=0 plane:

The displacement due to a dislocation along the y-axis (plane strain) can be written as:

$$u_i = \left[\frac{d_i}{2\pi}\cdot\arctan\left(\frac{z}{x}\right) - \epsilon_{ik}\cdot\frac{d_k}{2\pi}\cdot\ln\xi \right] + \frac{\alpha}{4\pi\mu\delta}\cdot\left[-(2\mu\epsilon_{ik}d_k)\cdot\ln\xi + (2\mu\epsilon_{nk}d_k)\cdot\delta\cdot x_n\cdot x_i\cdot\frac{1}{\xi^2} \right]$$

for i,k,n = 1,3

and: (1.5)

$$u_2 = 0$$

where: $$\epsilon_{ik} = \begin{cases} +1 & \text{for } i = 1,\ k = 3 \\ -1 & \text{for } i = 3,\ k = 1 \\ 0 & \text{otherwise} \end{cases}$$

d_1 and d_3 are the slip magnitude of the dislocations

We note that the terms in the second brackets expressing the displacements are of the form of line force expressions with equivalent magnitudes of $2\mu\epsilon_{ik}d_k$ and thus their Hansen potentials are already known. The Hansen potentials for the terms in the first bracket can be shown to be:

$$\varphi_1^{\text{first bracket}} = \frac{1}{2\pi}\cdot\left[d_1\cdot\left[z\cdot\ln\xi - z + x\cdot\arctan\left(\frac{z}{x}\right) \right] + d_3\cdot\left[z\cdot\arctan\left(\frac{z}{x}\right) - x\cdot\ln\xi + x \right] \right]$$

$$\varphi_2^{\substack{\text{first}\\\text{bracket}}} = 0 \qquad \varphi_3^{\substack{\text{first}\\\text{bracket}}} = 0 \tag{1.6}$$

Appendix 2: Derivation of the vector image method algorithm

The eigenfunction expansion method for elasticity problems in layered media was first formulated by Ben-Menahem and Singh 1968. We have used this method to derive the algorithm discussed in this paper. The notation (as far as possible) is the same as in the 1968 reference paper, although some new temporary terms have been defined in order to simplify the algebra for this specific implementation.

Any elastic displacement field satisfying the equilibrium equations:

$$\nabla^2 \underline{u} + (1+\lambda/\mu)\cdot\nabla\nabla\cdot\underline{u} = \underline{0} \tag{2.1}$$

can be written as the sum of $\underline{N}$, $\underline{F}$ and $\underline{M}$ (see (1)).

Using the method of the separation of variables in cylindrical coordinates on the potentials φ_1, φ_2 and φ_3 in the form:

$$\varphi = Z(z)\cdot R(r)\cdot F(\theta)$$

We get: (2.2)

$$\varphi = \exp(\pm kz)\cdot J_m(kr)\cdot\exp(\pm im\theta)$$

where J_m is the Bessel's function of the first kind and of m^{th} order.

Reexpressing φ_1, φ_2 and φ_3 in the above form and carrying out the ∇ and ∇ x and $\frac{\partial}{\partial z}$ operations in (1), we get:

$$\underline{u} = \sum_{m=1}^{\infty}\int_0^{\infty}\left[\, A_m^+\cdot N_m^+ + A_m^-\cdot N_m^- + B_m^+\cdot F_m^+ + B_m^-\cdot F_m^- + C_m^+\cdot M_m^+ + C_m^-\cdot M_m^- \,\right]\cdot dk \tag{2.3}$$

where: $A_m^{\pm}$, $B_m^{\pm}$ and $C_m^{\pm}$ are constant coefficients of $\underline{N}$, $\underline{F}$ and $\underline{M}$ dependent on 'm' only.

$$\underline{N}_m^{\pm} = \exp(\pm kz)\cdot\left[\ \pm\underline{P}_m + \underline{B}_m\ \right]$$

$$\underline{F}_m^{\pm} = \exp(\pm kz)\cdot\left[\ (\pm 1-2\delta kz)\cdot\underline{P}_m - (1\pm 2\delta kz)\cdot\underline{B}_m\ \right]$$

$$\underline{M}_m^{\pm} = \exp(\pm kz)\cdot\underline{C}_m$$

and:

$$\underline{P}_m = \hat{e}_z\cdot J_m(kr)\cdot\exp(im\theta)$$

$$\underline{B}_m = (\hat{e}_r\cdot\frac{\partial}{\partial kr} + \hat{e}_\theta\cdot\frac{1}{kr}\cdot\frac{\partial}{\partial\theta})J_m(kr)\cdot\exp(im\theta)$$

$$\underline{C}_m = (\hat{e}_r\cdot\frac{1}{kr}\cdot\frac{\partial}{\partial\theta}\cdot - \hat{e}_\theta\cdot\frac{\partial}{\partial kr})J_m(kr)\cdot\exp(im\theta) \tag{2.4}$$

In the above expressions for $\underline{P}_m$, $\underline{B}_m$ and $\underline{C}_m$ there is the implicit understanding that we can consider either the real or imaginary components of the expressions seperately.

From the above expressions for the displacements, the expressions for the tractions at a plane z=constant can be found, and we rewrite the above as:

$$\underline{u} = \sum_{m=0}^{\infty}\int_0^{\infty}(\underline{u}_m^R + \underline{u}_m^L)\cdot dk \tag{2.5}$$

$$\underline{T}^z = \sum_{m=0}^{\infty}\int_0^{\infty}(\underline{T}_m^R + \underline{T}_m^L)\cdot dk \tag{2.6}$$

where:

$$\underline{u}_m^R = x_m \cdot \underline{P}_m + y_m \cdot \underline{B}_m$$
$$\underline{T}_m^R = 2k \cdot X_m \cdot \underline{P}_m + 2k \cdot Y_m \cdot \underline{B}_m$$
$$\underline{u}_m^L = z_m \cdot \underline{C}_m$$
$$\underline{T}_m^L = k \cdot Z_m \cdot \underline{C}_m \tag{2.7}$$

and:

$$x_m = A_m^+ \cdot \exp(kz) - A_m^- \cdot \exp(-kz) + B_m^+ \cdot (1-2\delta kz) \cdot \exp(kz) + B_m^- \cdot (-1-2\delta kz) \cdot \exp(-kz)$$

$$y_m = A_m^+ \cdot \exp(kz) + A_m^- \cdot \exp(-kz) + B_m^+ \cdot (-1-2\delta kz) \cdot \exp(kz) + B_m^- \cdot (-1+2\delta kz) \cdot \exp(-kz)$$

$$z_m = C_m^+ \cdot \exp(kz) + C_m^- \cdot \exp(-kz)$$

$$X_m = A_m^+ \cdot \mu \cdot \exp(kz) + A_m^- \cdot \mu \cdot \exp(-kz) + B_m^+ \cdot \mu\delta \cdot (1-2kz) \cdot \exp(kz) + B_m^- \cdot \mu\delta \cdot (1+2kz) \cdot \exp(-kz)$$

$$Y_m = A_m^+ \cdot \mu \cdot \exp(kz) - A_m^- \cdot \mu \cdot \exp(-kz) + B_m^+ \cdot \mu\delta \cdot (-1-2kz) \cdot \exp(kz) + B_m^- \cdot \mu\delta \cdot (1-2kz) \cdot \exp(-kz)$$

$$Z_m = C_m^+ \cdot \mu \cdot \exp(kz) - C_m^- \cdot \mu \cdot \exp(-kz) \tag{2.8}$$

Notice that the $\underline{u}_m^R$ components are uncoupled from the $\underline{u}_m^L$ in the sense that the $A_m^{\pm}$, $B_m^{\pm}$ coefficients do not affect the $\underline{u}_m^L$ components and the $C_m^{\pm}$ coefficients do not affect the $\underline{u}_m^R$ components. We therefore treat the $\underline{u}_m^R$ and the $\underline{u}_m^L$ components seperately when analyzing a specific problem in terms of the Hansen potentials.

Consider the specific geometry shown in figure 2.1. The region consists of two elastic materials seperated by a planar interface. The material elasticity parameters used to characterize the regions are taken to be μ_1, δ_1 and μ_2, δ_2. A point source exists at the position z=-h. The problem is to find the displacement fields for region 1 (z<0) and for region 2 (z>0) under the influence of the point source, such that the displacements and the tractions are continuous across the interface plane (z=0).

In what follows, we are manipulating $\underline{u}_m^R$ and $\underline{u}_m^L$ in equation 2.5 for a fixed 'm', but the 'm' subscript will be dropped for brevity. First, we express the displacement and traction (on a z-plane) for (z+h) > 0 (which includes z=0) of a point source of arbitrary nature (using the eigenfunction expansion method and expressing the m'th component in matrix form) in the following way:

$$\begin{bmatrix} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{bmatrix}_0 = \left[\begin{array}{c|c} -1 & -1 - 2\delta_1 k\cdot(z+h) \\ \hline +1 & -1 + 2\delta_1 k\cdot(z+h) \\ \hline +2k\mu_1 & +2k\mu_1\delta_1\cdot[1+2k\cdot(z+h)] \\ \hline -2k\mu_1 & +2k\mu_1\delta_1\cdot[1-2k\cdot(z+h)] \end{array}\right] \cdot \begin{bmatrix} A^{0-} \\ \hline B^{0-} \end{bmatrix} \cdot \exp(-k|z+h|)$$

$$\begin{bmatrix} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{bmatrix}_0 = \begin{bmatrix} +1 \\ \hline -k\mu_1 \end{bmatrix} \begin{bmatrix} C^{0-} \end{bmatrix} \cdot \exp(-k|z+h|)$$

(2.9)

and we define:

$$\left[\begin{array}{c} S_1 \\ \hline S_2 \\ \hline S_3 \\ \hline S_4 \end{array}\right] \equiv -1 \cdot \left.\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{array}\right]_0 \right|_{z=0} \qquad \left[\begin{array}{c} S_5 \\ \hline S_6 \end{array}\right] = -1 \cdot \left.\left[\begin{array}{c} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{array}\right]_0 \right|_{z=0} \tag{2.10}$$

Now the elastic fields in region 1 are expressible as:

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{array}\right]_1 = \left[\begin{array}{c|c} +1 & +1 - 2\delta_1 k \cdot z \\ \hline +1 & -1 - 2\delta_1 k \cdot z \\ \hline +2k\mu_1 & +2k\mu_1\delta_1 \cdot (1-2kz) \\ \hline +2k\mu_1 & +2k\mu_1\delta_1 \cdot (-1-2kz) \end{array}\right] \cdot \left[\begin{array}{c} A^{1+} \\ \hline B^{1+} \end{array}\right] \cdot \exp(kz)$$

$$+ \left[\begin{array}{c} \text{terms due to the} \\ \text{point source} \\ \text{as given above} \end{array}\right]$$

$$\left[\begin{array}{c} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{array}\right]_1 = \left[\begin{array}{c} +1 \\ \hline +k\mu_1 \end{array}\right] \left[C^{1+} \right] \cdot \exp(kz) + \left[\begin{array}{c} \text{terms due to the} \\ \text{point source} \\ \text{as given above} \end{array}\right] \tag{2.11}$$

And the elastic fields in region 2 are expressible as:

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{array}\right]_2 = \left[\begin{array}{c|c} -1 & -1 - 2\delta_2 kz \\ \hline +1 & -1 + 2\delta_2 kz \\ \hline +2k\mu_2 & +2k\mu_2\delta_2 \cdot (1+2kz) \\ \hline -2k\mu_2 & +2k\mu_2\delta_2 \cdot (1-2kz) \end{array}\right] \cdot \left[\begin{array}{c} A^{2-} \\ \hline B^{2-} \end{array}\right] \cdot \exp(-kz)$$

$$\begin{bmatrix} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{bmatrix}_2 = \begin{bmatrix} +1 \\ \hline -k\mu_2 \end{bmatrix} \begin{bmatrix} C^{2-} \end{bmatrix} \cdot \exp(-kz) \tag{2.12}$$

Applying the condition that $\underline{u}_m$ and $\underline{T}_m$ are to be continuous (for each m) along the interface plane z=0, we get:

$$\left[\begin{array}{c|c|c|c} +1 & +1 & +1 & +1 \\ \hline +1 & -1 & -1 & +1 \\ \hline +2k\mu_1 & +2k\mu_1\delta_1 & -2k\mu_2 & -2k\mu_2\delta_2 \\ \hline +2k\mu_1 & -2k\mu_1\delta_1 & +2k\mu_2 & -2k\mu_2\delta_2 \end{array}\right] \cdot \begin{bmatrix} A^{1+} \\ \hline B^{1+} \\ \hline A^{2-} \\ \hline B^{2-} \end{bmatrix} = \begin{bmatrix} S_1 \\ \hline S_2 \\ \hline S_3 \\ \hline S_4 \end{bmatrix}$$

$$\left[\begin{array}{c|c} +1 & -1 \\ \hline +k\mu_1 & +k\mu_2 \end{array}\right] \cdot \begin{bmatrix} C^{1+} \\ \hline C^{2-} \end{bmatrix} = \begin{bmatrix} S_5 \\ \hline S_6 \end{bmatrix} \tag{2.13}$$

Now we solve for A^{1+}, B^{1+}, C^{1+} and A^{2-}, B^{2-}, C^{2-} by inverting the 4x4 and 2x2 system of equations. We obtain:

$$\begin{bmatrix} A^{1+} \\ \hline B^{1+} \\ \hline A^{2-} \\ \hline B^{2-} \end{bmatrix} = \frac{1}{\Delta} \cdot \left[\begin{array}{c|c|c|c} +\Delta/2-\gamma-\delta_1 & +\Delta/2-\gamma-\delta_1 & +\frac{\gamma+\delta_1}{2k\mu_1} & +\frac{\gamma+\delta_1}{2k\mu_1} \\ \hline +\Delta/2-\gamma\delta_1\delta_2-\delta_1 & -\Delta/2+\gamma\delta_1\delta_2+\delta_1 & +\frac{\gamma\delta_2+1}{2k\mu_1} & -\frac{\gamma\delta_2+1}{2k\mu_1} \\ \hline +\gamma\delta_1\delta_2+\delta_1 & -\gamma\delta_1\delta_2-\delta_1 & -\frac{\gamma\delta_2+1}{2k\mu_1} & +\frac{\gamma\delta_2+1}{2k\mu_1} \\ \hline +\gamma+\delta_1 & +\gamma+\delta_1 & -\frac{\gamma+\delta_1}{2k\mu_1} & -\frac{\gamma+\delta_1}{2k\mu_1} \end{array}\right] \cdot \begin{bmatrix} S_1 \\ \hline S_2 \\ \hline S_3 \\ \hline S_4 \end{bmatrix}$$

$$\begin{bmatrix} C^{1+} \\ \hline C^{2-} \end{bmatrix} = \frac{1}{\gamma+1} \cdot \left[\begin{array}{c|c} +\gamma & +\frac{1}{k\mu_1} \\ \hline -1 & +\frac{1}{k\mu_1} \end{array}\right] \cdot \begin{bmatrix} S_5 \\ \hline S_6 \end{bmatrix} \tag{2.14}$$

where: $\gamma \equiv \mu_2/\mu_1$

$$\Delta \equiv 2\cdot(\gamma+\delta_1)\cdot(\gamma\delta_2+1)$$

Expressing the S's in terms of A^{0-}, B^{0-} and C^{0-}, and simplifying the expressions we get:

$$\begin{bmatrix} A^{1+} \\ \hline B^{1+} \\ \hline A^{2-} \\ \hline B^{2-} \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1-b \\ \hline 1-a & 2\delta_1\cdot(1-a)\cdot kh \\ \hline a & 2\delta_1 a\cdot kh \\ \hline 0 & b \end{array}\right]\cdot\begin{bmatrix} A^{0-} \\ \hline B^{0-} \end{bmatrix}\cdot\exp(-kh)$$

$$\begin{bmatrix} C^{1+} \\ \hline C^{2-} \end{bmatrix} = \begin{bmatrix} \dfrac{1-\gamma}{1+\gamma} \\ \hline \dfrac{2}{1+\gamma} \end{bmatrix}\cdot\begin{bmatrix} C^{0-} \end{bmatrix}\cdot\exp(-kh) \tag{2.15}$$

where:

$$a \equiv (\delta_1+1)/(\gamma+\delta_1)$$

$$b \equiv (\delta_1+1)/(\gamma\cdot\delta_2+1) \tag{2.16}$$

Therefore we find that the displacement field (for a given m) in region 1 and region 2 can be written as:

$$\begin{bmatrix} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{bmatrix}_1 = \left[\begin{array}{c|c} +1 & +1-2\delta_1 kz \\ \hline +1 & -1-2\delta_1 kz \end{array}\right]\cdot\left[\begin{array}{c|c} 0 & 1-b \\ \hline 1-a & 2\delta_1\cdot(1-a)\cdot kh \end{array}\right]\cdot\begin{bmatrix} A^{0-} \\ \hline B^{0-} \end{bmatrix}\cdot\exp[k(z-h)]$$

$$+\begin{bmatrix} \text{source terms} \end{bmatrix}$$

$$\left[\ \underline{u}(\underline{C})\ \right]_1 = \left[\ +1\ \right]\cdot\left[\ \frac{1-\gamma}{1+\gamma}\ \right]\cdot\left[\ C^{0-}\right]\cdot\exp[k(z-h)] \quad + \left[\ \text{source terms}\ \right] \tag{2.17}$$

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_2 = \left[\begin{array}{c|c} -1 & -1-2\delta_2 kz \\ \hline +1 & -1+2\delta_2 kz \end{array}\right]\cdot\left[\begin{array}{c|c} +a & 2\delta_1 a\cdot kh \\ \hline 0 & +b \end{array}\right]\cdot\left[\begin{array}{c} A^{0-} \\ \hline B^{0-} \end{array}\right]\cdot\exp[-k(z+h)]$$

$$\left[\ \underline{u}(\underline{C})\ \right]_2 = \left[\ +1\ \right]\cdot\left[\ \frac{2}{1+\gamma}\ \right]\cdot\left[\ C^{0-}\right]\cdot\exp[-k(z+h)] \tag{2.18}$$

The $\underline{u}(\underline{C})$ terms are in a form from which we can deduce the algorithm, however, the $\underline{u}(\underline{P})$ and $\underline{u}(\underline{B})$ terms have to be further manipulated. We now try to express the $\underline{u}(\underline{P})$ and $\underline{u}(\underline{B})$ terms in the following manner:

$$\begin{aligned}
\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_1 = {} & \left[\begin{array}{c|c} +1 & +1-2\delta_1 k\cdot(z-h) \\ \hline +1 & -1-2\delta_1 k\cdot(z-h) \end{array}\right]\cdot\left[\begin{array}{c} A_a^+ \\ \hline B_a^+ \end{array}\right]\cdot\exp[k(z-h)] \\
& + \frac{\partial}{\partial z}\left[\begin{array}{c|c} +1 & +1-2\delta_1 k\cdot(z-h) \\ \hline +1 & -1-2\delta_1 k\cdot(z-h) \end{array}\right]\cdot\left[\begin{array}{c} A_b^+ \\ \hline B_b^+ \end{array}\right]\cdot\exp[k(z-h)] \\
& + \frac{\partial^2}{\partial z^2}\left[\begin{array}{c} +1 \\ \hline +1 \end{array}\right]\cdot\left[\ A_c^+\ \right]\cdot\exp[k(z-h)] \quad + \left[\ \text{source terms}\ \right]
\end{aligned} \tag{2.19}$$

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_2 = \left[\begin{array}{c|c} -1 & -1-2\delta_2 k\cdot(z+h) \\ \hline +1 & -1+2\delta_2 k\cdot(z+h) \end{array}\right]\cdot\left[\begin{array}{c} A_a^- \\ \hline B_a^- \end{array}\right]\cdot\exp[-k(z+h)]$$

$$+\frac{\partial}{\partial z}\left[\begin{array}{c|c} -1 & -1-2\delta_2 k\cdot(z+h) \\ \hline +1 & -1+2\delta_2 k\cdot(z+h) \end{array}\right]\cdot\left[\begin{array}{c} A_b^- \\ \hline B_b^- \end{array}\right]\cdot\exp[-k(z+h)]$$

$$+\frac{\partial^2}{\partial z^2}\left[\begin{array}{c} -1 \\ \hline +1 \end{array}\right]\cdot\left[\; A_c^- \;\right]\cdot\exp[-k(z+h)] \tag{2.20}$$

Noting that:

$$\frac{\partial^n}{\partial z^n}\exp[k(z-h)] = k^n\cdot\exp[k(z-h)]$$

and (2.21)

$$\frac{\partial^n}{\partial z^n}\exp[-k(z+h)] = (-k)^n\cdot\exp[-k(z+h)]$$

We obtain:

$$A_a^+ = (1-b)\cdot B^{0-} \qquad B_a^+ = (1-a)\cdot A^{0-}$$

$$A_b^+ = -2\delta_1\cdot(1-a)\cdot h\cdot A^{0-} + 4\delta_1^2\cdot(1-a)\cdot h\cdot B^{0-} \tag{2.22}$$

$$B_b^+ = 2\delta_1\cdot(1-a)\cdot h\cdot B^{0-} \qquad A_c^+ = -4\delta_1^2\cdot(1-a)\cdot h^2\cdot B^{0-}$$

and:

$$A_a^- = a\cdot A^{0-} \qquad B_a^- = b\cdot B^{0-}$$

$$A_b^- = 2\cdot(\delta_2 b-\delta_1 a)\cdot h\cdot B^{0-} \qquad B_b^- = A_c^- = 0 \tag{2.23}$$

Since the above relations are true for each component of a potential, then they must be true for the whole potential and we get:

if:

$$\underline{u}^0 = \underline{N}(-h,\varphi_1^0) + \underline{F}(\delta_1,-h,\varphi_2^0) + \underline{M}(-h,\varphi_3^0)$$

then:

$$\begin{aligned}\underline{u}^1 = \underline{u}^0 &+ \underline{N}(h,(1-b)\cdot\overline{\varphi}_2^0) + \frac{\partial}{\partial z}\underline{N}(h,-2\delta_1\cdot(1-a)\cdot h\cdot\overline{\varphi}_1^0)\\ &+ \frac{\partial}{\partial z}\underline{N}(h,+4\delta_1^2\cdot(1-a)\cdot h\cdot\overline{\varphi}_2^0)\\ &+ \frac{\partial^2}{\partial z^2}\underline{N}(h,-4\delta_1^2\cdot(1-a)\cdot h^2\cdot\overline{\varphi}_2^0)\\ &+ \underline{F}(\delta_1,h,(1-a)\overline{\varphi}_1^0)\\ &+ \frac{\partial}{\partial z}\underline{F}(\delta_1,h,2\delta_1\cdot(1-a)\cdot h\cdot\overline{\varphi}_2^0)\\ &+ \underline{M}(h,\frac{1-\gamma}{1+\gamma}\cdot\overline{\varphi}_3^0)\end{aligned} \tag{2.24}$$

$$\begin{aligned}\underline{u}^2 = &\ \underline{N}(-h,a\varphi_1^0) + \frac{\partial}{\partial z}\underline{N}(-h,2\cdot(\delta_2 b-\delta_1 a)\cdot h\cdot\varphi_2^0)\\ &+ \underline{F}(\delta_2,-h,b\varphi_2^0)\\ &+ \underline{M}(-h,\frac{2}{1+\gamma}\cdot\varphi_3^0)\end{aligned} \tag{2.25}$$

Noting that:

$$\frac{\partial^n}{\partial z^n}\underline{N}(h,cst\cdot\varphi) = \underline{N}(h,cst\cdot\frac{\partial^n}{\partial z^n}\varphi)$$

and (2.26)

$$\frac{\partial}{\partial z}F(\delta,h,cst\cdot\varphi) = N(h,-2\delta_1\cdot cst\cdot\frac{\partial}{\partial z}\varphi) + F(\delta,h,cst\cdot\frac{\partial}{\partial z}\varphi)$$

where: cst is a constant

We obtain the algorithm given in the main body of this paper (**two minor differences are: i) The statement of the algorithm in** the paper considers region 1 to be at $z>0$ and hence $z=+h$ instead of $z=-h$ to be the location of the source point and ii) A formalism in terms of matrix operators is implemented in the main text).

Appendix 3: Derivation of some sample Green's functions through the use of the image algorithm

In this section, we consider displacement fields in Cartesian components for some sample point source problems in a halfspace with a free surface. The solutions that will be rederived are readily available (and established) in the literature and hence serve as an empirical check of the algorithm. In addition, these specific examples help clarify details of the application of the algorithm. From the algorithm presented in the main text and from the examples to follow, it should be clear that many established elastostatic solutions and new solutions (in part II of this paper) can be obtained in a systematic fashion.

By considering a halfspace problem with a free surface, we obtain the following simplifications:

$$\gamma = 0 \qquad 1-a = -1/\delta_1 \qquad 1-b = -\delta_1$$

call:

$$\delta \equiv \delta_1$$

then:

$$\underline{\underline{R}}_R(h,a,b,\delta_1) = \left[\begin{array}{c|c} \dfrac{-2h\cdot\frac{\partial}{\partial z}}{-1/\delta} & \dfrac{-\delta + 4\delta h^2\cdot\frac{\partial^2}{\partial^2}}{+2h\cdot\frac{\partial}{\partial z}} \end{array}\right] \qquad \underline{R}_L(\gamma) = \left[\ 1\ \right] \tag{3.1}$$

The following example problems will be considered:

I. Screw dislocation (Antiplane problem).

II. Point force acting interior to the halfspace with a free surface (Mindlin's solution).

  i) The point force is in the x direction.

  ii) The point force is in the z direction.

I. Screw dislocation (antiplane problem)

For the antiplane problem, all that is required is to obtain the image potential with respect to the interface plane since the $\underline{\underline{R}}_L$ matrix is the identity operator. Also, we notice that getting the image of a given $\underline{M}$ type (see main text, particularly equation 8, and with specialization in 3.1) displacement field is equal to the $\underline{M}$ displacement field of the image potential describing that field (i.e. $\overline{\underline{M}(h,\underline{r})} = \underline{M}(-h,\overline{\underline{r}})$), and hence we can directly operate on a given displacement field when using the algorithm for a purely antiplane problem. This corresponds to the scalar field image method for the antiplane case.

As an example we consider the field due to a screw dislocation in the plane perpendicular to the x-z plane at location z=h and x=0. The field due to the dislocation in infinite space is:

$$u_y = \arctan[(z-h)/x] \qquad (3.2)$$

The image field will be $\overline{u}_y$ which implies that the combined fields give:

$$u_y = \arctan[(z-h)/x] - \arctan[(z+h)/x] \qquad (3.3)$$

Of course this is but a simple application of the scalar image method.

II. Point force acting interior to a halfspace with a free surface.

For this problem the potentials for the point source in infinite space are given in appendix 1. However, we have a choice of where to locate the singularities of the potentials. Since we do not want the image potentials to introduce any new sources inside the halfspace, we choose the infinite space potentials to have all their singularities in that halfspace. The following functions will need to be calculated:

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\bar{\varphi}_2^0\right] \qquad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial z^2}\right] \cdot \left[\bar{\varphi}_1^0\right]$$

$$\frac{\partial}{\partial z} \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\bar{\varphi}_1^0\right] \qquad \frac{\partial^2}{\partial z^2} \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\bar{\varphi}_2^0\right]$$

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial z^2}\right] \cdot \left[\bar{\varphi}_2^0\right]$$

There are some repetition in the suggested functions to be calculated since the partial differentiation operations are commutative when the function is sufficiently smooth.

We note that the potentials to be differentiated are linear combinations of the following functions:

$$\bar{\varphi}_A = x/[2 \cdot (r+z)]$$
$$\bar{\varphi}_C = -[\ln(r+z)]/2 \tag{3.4}$$

In addition, we have an antiplane potential for this case which is a linear multiple of the following function:

$$\bar{\varphi}_B = y/[2 \cdot (r+z)] \tag{3.5}$$

and we will also have to calculate:

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] \cdot \left[\bar{\varphi}_B\right]$$

Now we can perform the required differentiations and we also note the following:

$$(x^2z)/[r^3\cdot(r+z)] + (x^2z)/[r^2\cdot(r+z)^2] = x^2/r^3 - x^2/[r\cdot(r+z)^2]$$
$$z/[r\cdot(r+z)] = 1/r - 1/(r+z)$$
$$y^2/[r\cdot(r+z)^2] = 2/(r+z) - 1/r - x^2/[r\cdot(r+z)^2]$$
$$(xyz)/[r^3\cdot(r+z)] + (xyz)/[r^2\cdot(r+z)^2)] = (xy)/r^3 - (xy)/[r\cdot(r+z)^2]$$
$$-\delta/2+1/(2\delta) = \mu\cdot(1+\delta)/(\lambda+\mu) \qquad (3.6)$$

i) Case when the force is acting in the x direction:

Defining:

$$r_2^2 = x^2 + y^2 + (z+h)^2$$

We get after simplification and the use of identities similar to those given in (3.6), but with z replaced by (z+h) and "r" replaced by "r_2" wherever they occur:

$$\begin{aligned}
\underline{u} = \underline{u}^0 \\
&+ [1/4\pi\mu(1+\delta)]\cdot\Big[\\
&\hat{e}_x\cdot\Big[[\delta/2 + 1/(2\delta) + 1 - 2\cdot(1+\delta) + (1+\delta)]/(r_2+z+h) \\
&\qquad + [-\delta/2 - 1/(2\delta) - 1 + (1+\delta)]\cdot x^2/[r_2\cdot(r_2+z+h)^2] \\
&\qquad + [-1 + (1+\delta)]/r_2 \\
&\qquad + x^2/r_2^3 + 2\delta hz\cdot[1/r_2^3 - 3\cdot x^2/r_2^5] \Big] \\
&+ \hat{e}_y\cdot\Big[[-\delta/2 - 1/(2\delta) - 1 + (1+\delta)]\cdot(xy)/[r_2\cdot(r_2+z+h)^2] \\
&\qquad + (xy)/r_2^3 + 2\delta hz\cdot[-3\cdot x\cdot y/r_2^5] \Big] \\
&+ \hat{e}_z\cdot\Big[[-\delta/2 + 1/(2\delta)]\cdot x/[r_2\cdot(r_2+z+h)] \\
&\qquad + (z-h)\cdot x/r_2^3 - 6\delta hz\cdot x\cdot(z+h)/r_2^5 \Big] \Big]
\end{aligned} \qquad (3.7)$$

Noting that:

$$\delta/[4\pi\mu\cdot(1+\delta)] = 1/[16\pi\mu\cdot(1-\nu)]$$
$$1/\delta = 3-4\nu \qquad (3.8)$$
$$-\delta/2+1/(2\delta) = 4\delta\cdot(1-\nu)\cdot(1-2\nu)$$

we find that the above result (3.7) coincides with the solution first obtained by Mindlin (1936) and shown in Mura (1982).

ii) Case when the force is acting in the z direction:

After simplifications we get:

$$\begin{aligned}
\underline{u} = \underline{u}^0 & \\
& + [1/4\pi\mu(1+\delta)]\cdot\Bigg[\\
& \hat{e}_x\cdot\Big[+ [\delta/2 - 1/(2\delta)]\cdot x/[r_2\cdot(r_2+z+h)] \\
& \qquad + (z-h)\cdot x/r_2^3 + 6\delta hz\cdot x\cdot(z+h)/r_2^5 \Big] \\
& + \hat{e}_y\cdot\Big[[\delta/2 - 1/(2\delta)]\cdot y/[r_2\cdot(r_2+z+h)] \\
& \qquad + (z-h)\cdot y/r_2^3 + 6\delta hz\cdot y\cdot(z+h)/r_2^5 \Big] \\
& + \hat{e}_z\cdot\Big[[\delta/2 + 1/(2\delta)]/r_2 \\
& \qquad + [(z+h)^2 - 2\delta hz]/r_2^3 + 6\delta hz\cdot(z+h)^2/r_2^5 \Big] \Bigg]
\end{aligned} \qquad (3.9)$$

Noting the relations given in (3.8) and:

$$\delta/2+1/(2\delta) = \delta\cdot[8\cdot(1-\nu)^2 - (3-4\nu)] \qquad (3.10)$$

we find that the above result (3.9) coincides with the solution first obtained by Mindlin (1936) and shown in Mura (1982).

Figure Captions

figure 1:
2 bonded elastic halfspaces with a point source at z=h.

figure 2.1:
2 bonded elastic halfspaces with a point source at z=-h.

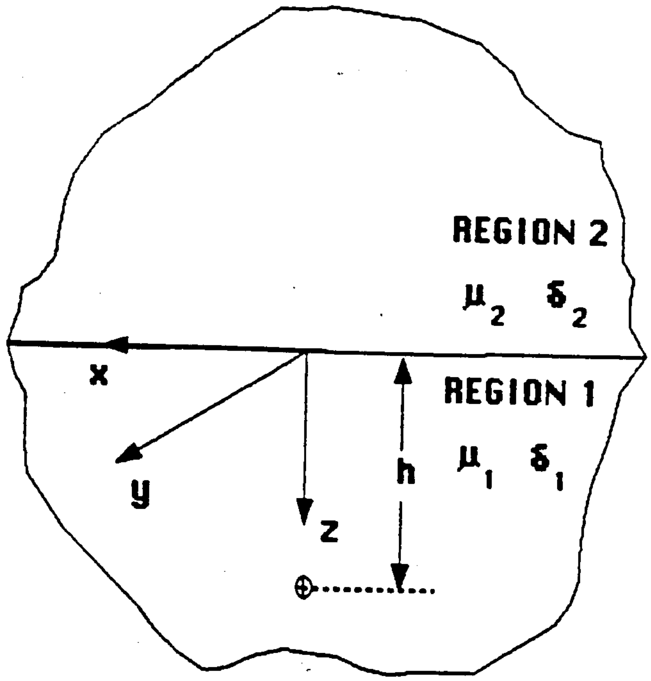

Figure 1

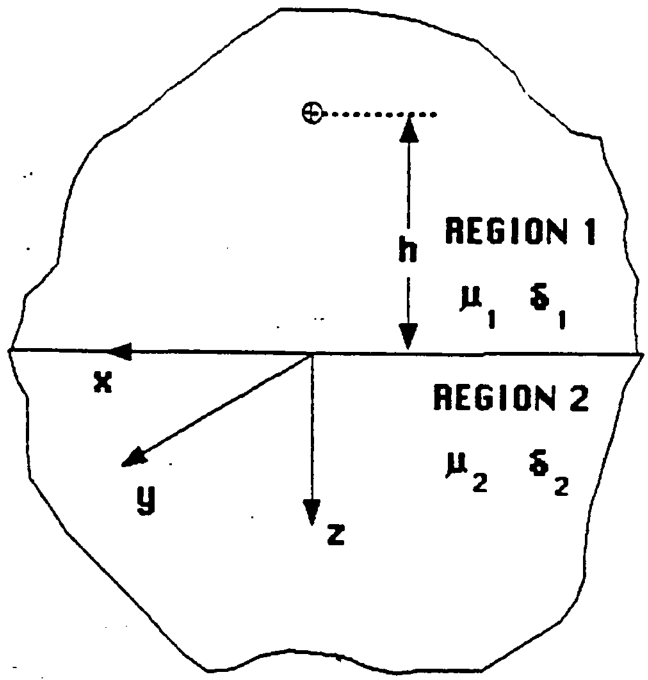

Figure 2.1